Looking at distribution of finite orbits

Recall **Thm** Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rotation by an angle α, then either

(a) every orbit has period q where ($\alpha = \frac{p}{q}$ in reduced form)

(b) every orbit is dense ($\alpha \notin \mathbb{Q}$)

How to build rational rotations with orbits being less and less sparse.

Naive approach

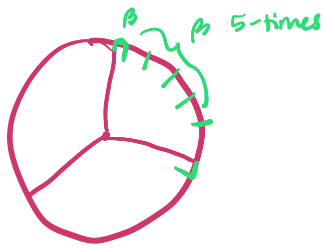
$\alpha = \frac{1}{n}$ not helpful :(

Issue: the limiting rotation # is $\alpha = 0$ which is uninteresting

Want: something irrational in limit?

Another more interesting example:

Start with $R_{1/3}$



the gap between 0 and 3α is β so

$$\beta = 1 - 3\alpha$$

$$5\beta = \alpha$$

now can solve!

$$\frac{\alpha}{5} = 1 - 3\alpha$$

$$3\alpha + \frac{\alpha}{5} = 1$$

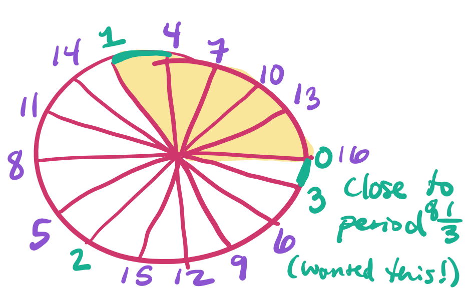
$$\alpha\left(3 + \frac{1}{5}\right) = 1$$

$$\alpha = \frac{1}{3 + \frac{1}{5}}$$

first step of a continued fraction

$$= \frac{5}{16}$$

Check



3 steps to come back to wedge
and each step is like β

1^{st} return map to $[0,\alpha)$:
If $x \in [0,\alpha)$ define
$F(x)$ = 1^{st} time the forward orbit of x revisits $[0,\alpha)$.

In the example, if the wedge is streched into a circle, the 1^{st} return map is another rotation by angle $\frac{1}{5}$

Exercise 1)

find equations explaining why we can think of the first return of $\alpha = \frac{1}{a_1 + \frac{1}{a_2}}$ to the interval $[0,\alpha)$ as a rotation by $\frac{1}{a_2}$.

We can iterate this procedure!

F = 1^{st} return map is rotation by $\frac{1}{5}$

Consider 1^{st} return of this region

gap γ

Go around 4 times to get periodic

$\frac{1}{5} \rightarrow \frac{1}{5+\frac{1}{4}}$ new α is ...

$$\alpha = \cfrac{1}{3+\cfrac{1}{5+\frac{1}{4}}}$$

First return maps

Def let $f: X \to X$ be a dynamical system and $A \subset X$
(our wedge before)

The first return dynamics to A is the dynamical system

$$F: A \to A \quad \text{such that}$$

$F(x) =$ the first positive iterate $f^k(x)$ such that $f^k(x) \in A$.

Exercise 2)

Prove that the first return map of R_α for the interval of $[0, \alpha)$ is conjugated to the rotation by angle $\frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor$

Exercise 0)

Draw some orbits and connect to the lecture $\alpha = \cfrac{1}{2+\frac{1}{3}} = \frac{3}{7}$

and $\alpha = \cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}} = \frac{5}{14}$